

MATHEMATICS



Vector Algebra

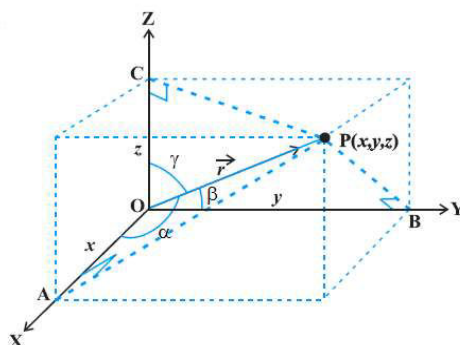
Top Concepts

1. A quantity which has magnitude as well as direction is called a **vector**.
2. A directed line segment is called a vector.



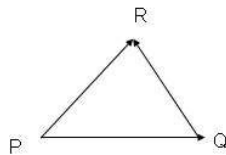
The point X from where the vector starts is called the initial point and the point Y where it ends is called the terminal point.

3. For vector \overline{XY} , magnitude = distance between X and Y, denoted by $|\overline{XY}|$, and is greater than or equal to zero.
4. The distance between the initial point and the terminal point is called the magnitude of the vector.
5. The position vector of point $P \equiv (x_1, y_1, z_1)$ with respect to the origin is given by $\overline{OP} = \vec{r} = \sqrt{x_1^2 + y_1^2 + z_1^2}$.
6. If the position vector \overline{OP} of a point P makes angles α , β and γ with the x, y and z axes, respectively, then α , β and γ are called the **direction angles** and $\cos \alpha$, $\cos \beta$ and $\cos \gamma$ are called the **direction cosines** of the position vector \overline{OP} .
7. $\lambda = \cos \alpha$, $m = \cos \beta$ and $n = \cos \gamma$ are called the direction cosines of \vec{r} .



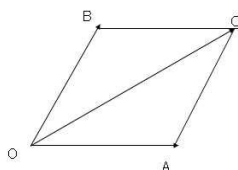
8. The numbers l , m and n , proportional to l , m and n , are called the direction ratios of the vector \vec{r} and are denoted by a , b and c .
In general, $l^2 + m^2 + n^2 = 1$, but $a^2 + b^2 + c^2 \neq 1$.
9. Vectors can be classified on the basis of position and magnitude. On the basis of magnitude, vectors are classified as zero vectors and unit vectors. On the basis of position, vectors are classified as co-initial vectors, parallel vectors, free vectors and collinear vectors.
10. A zero vector is a vector whose initial and terminal points coincide and is denoted by $\vec{0}$. $\vec{0}$ is called the additive identity.
11. A unit vector has a magnitude equal to 1. A unit vector in the direction of the given vector \vec{a} is denoted by \hat{a} .
12. For a given vector \vec{a} , the vector $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$ gives the unit vector in the direction of \vec{a} .
13. Co-initial vectors are vectors with the same initial point.
14. Collinear vectors are parallel to the same line irrespective of their magnitudes and directions.
15. Two vectors are said to be parallel if they are non-zero scalar multiples of one another.
16. Equal vectors, as the name suggests, are vectors which have the same magnitude and direction irrespective of their initial points.
17. The negative vector of a given vector \vec{a} is a vector which has the same magnitude as \vec{a} but its direction is the opposite of \vec{a} .
18. A system of vectors is said to be coplanar, if the vectors lie in the same plane.
19. Reciprocal of a vector: A vector with the same direction as that of the given vector \vec{a} but magnitude equal to the reciprocal of the given vector is known as the reciprocal of \vec{a} and is denoted by $\overline{a^{-1}}$.
Thus, if \vec{a} is a vector with magnitude a , then $|\overline{a^{-1}}| = \frac{1}{a}$.
20. A vector whose initial position is not fixed is called a free vector.
21. Two vectors can be added using the triangle law and parallelogram law of vector addition.

22. **Triangle Law of Vector Addition:** Suppose two vectors are represented by two sides of a triangle in sequence, then the third closing side of the triangle represents the sum of the two vectors.



$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

23. **Parallelogram Law of Vector Addition:** If two vectors \vec{a} and \vec{b} are represented by two adjacent sides of a parallelogram in magnitude and direction, then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by the diagonal of the parallelogram.



$$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$$

24. Vector addition is both commutative as well as associative.

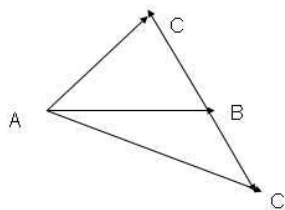
Thus, (i) Commutative: For any two vectors \vec{a} and \vec{b} , we have

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

(ii) Associativity: For any three vectors, \vec{a} , \vec{b} , and \vec{c} , we have

$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$$

25. **Difference of vectors:** To subtract a vector \overrightarrow{BC} from vector \overrightarrow{AB} , its negative is added to \overrightarrow{AB} .



$$\overrightarrow{BC}' = -\overrightarrow{BC}$$

$$\overrightarrow{AB} + \overrightarrow{BC}' = \overrightarrow{AC}'$$

$$\Rightarrow \overrightarrow{AB} - \overrightarrow{BC} = \overrightarrow{AC}'$$

26. For any two vectors \vec{a} and \vec{b} , we have $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$.

27. For any two vectors \vec{a} and \vec{b} , we have $|\vec{a} - \vec{b}| \leq |\vec{a}| + |\vec{b}|$.

28. For any two vectors \vec{a} and \vec{b} , we have $|\vec{a} - \vec{b}| \geq \left| |\vec{a}| - |\vec{b}| \right|$.

29. If \vec{a} is any vector and k is any scalar, then the scalar product of \vec{a} and k is $k\vec{a}$.

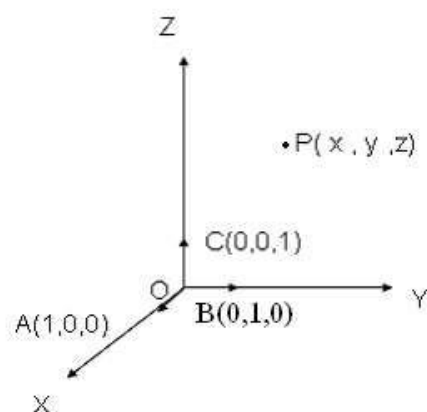
$k\vec{a}$ is also a vector, collinear to the vector \vec{a} .

$k > 0 \Rightarrow k\vec{a}$ has the same direction as \vec{a} .

$k < 0 \Rightarrow k\vec{a}$ has the opposite direction as \vec{a} .

Magnitude of $k\vec{a}$ is $|k|$ times the magnitude of vector $k\vec{a}$.

30. Unit vectors along OX, OY and OZ are denoted by \hat{i} , \hat{j} and \hat{k} , respectively.



Vector $\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is called the component form of vector r .

Here, x , y and z are called the scalar components of \vec{r} in the directions of \hat{i} , \hat{j} and \hat{k} , and

$x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are called the vector components of the vector r along the respective axes.

31. Two vectors \vec{a} and \vec{b} are collinear $\Leftrightarrow \vec{b} = k\vec{a}$, where k is a non-zero scalar. Vectors \vec{a} and $k\vec{a}$ are always collinear.

32. If \vec{a} and \vec{b} are equal, then $|\vec{a}| = |\vec{b}|$.

33. If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are any two points, then the vector joining P and Q is \overrightarrow{PQ} = position vector of Q – position vector of P , i.e. $\overrightarrow{PQ} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$.

34. Components of a vector in two dimensions: If Q is a point $X(a, b)$, then

(i) $\overrightarrow{OQ} = a\hat{i} + b\hat{j}$

(ii) $|\overrightarrow{OQ}| = \sqrt{a^2 + b^2}$

(iii) The components of \overrightarrow{OQ} along the x -axis is a vector $a\hat{i}$ whose magnitude is $|a|$ and whose direction is along OX or OX' as a is positive or negative.

(iv) The components of \overrightarrow{OQ} along the y -axis is a vector $b\hat{j}$ whose magnitude is $|b|$ and whose direction is along OY or OY' as b is positive or negative.

35. Components of a vector in three dimensions: If Q is a point X(a, b, c), then

(i) $\vec{OQ} = a\hat{i} + b\hat{j} + c\hat{k}$

(ii) $|\vec{OQ}| = \sqrt{a^2 + b^2 + c^2}$

(iii) The components of \vec{OQ} along the x-axis is a vector $a\hat{i}$ whose magnitude is $|a|$ and whose direction is along OX or OX' as a is positive or negative.

(iv) The components of \vec{OQ} along the y-axis is a vector $b\hat{j}$ whose magnitude is $|b|$ and whose direction is along OY or OY' as b is positive or negative.

(v) The components of \vec{OQ} along the z-axis is a vector $c\hat{k}$ whose magnitude is $|c|$ and whose direction is along OZ or OZ' as c is positive or negative.

36. Collinearity of vectors: If \vec{a} and \vec{b} are two collinear or parallel vectors, then there exists a scalar k such that $\vec{a} = k\vec{b}$.

37. Two non-zero vectors \vec{a} and \vec{b} are collinear if and only if there exists scalars x and y, not both zero, such that $x\vec{a} + y\vec{b} = \vec{0}$.

38. If \vec{a} and \vec{b} are any two non-zero non-collinear vectors and x and y are scalars, then $x\vec{a} + y\vec{b} = \vec{0} \Rightarrow x = y = 0$.

39. Three points with position vectors \vec{a}, \vec{b} and \vec{c} are collinear if and only if there exists three scalars x, y and z, not all zero simultaneously, such that $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ and $x + y + z = 0$.

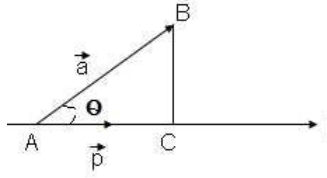
40. Let \vec{a} and \vec{b} be two given non-zero non-collinear vectors. Then any vector \vec{r} coplanar with \vec{a} and \vec{b} can be uniquely expressed as $\vec{r} = m\vec{a} + n\vec{b}$ for scalars m and n.

41. Three vectors \vec{a}, \vec{b} and \vec{c} are coplanar if and only if $l\vec{a} + m\vec{b} + n\vec{c} = \vec{0}$, where l, m and n are scalars not all zero simultaneously.

42. If \vec{a}, \vec{b} and \vec{c} are any three non-zero non-coplanar vectors, and x, y and z are scalars, then $x\vec{a} + y\vec{b} + z\vec{c} = \vec{0} \Rightarrow x = y = z = 0$.

43. Geometrical interpretation of the scalar product: scalar product of two vectors \vec{a} and \vec{b} is the product of modulus of either \vec{a} or \vec{b} , and the projection of the other in its direction.

44. Projection of vector **AB**, making an angle of θ with the line L, on line L is vector $\vec{P} = |\overline{AB}| \cos \theta$.



45. Properties of a scalar product:

(i) Scalar product is commutative, i.e. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

(ii) Distributive property of a scalar product over addition:

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$$

$$(\vec{b} + \vec{c}) \cdot \vec{a} = \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a}$$

(iii) If the scalar product $\vec{a} \cdot \vec{b} = 0$, then \vec{a} and \vec{b} are perpendicular to each other.

(iv) If \vec{a} and \vec{b} are perpendicular to each other, then the scalar product $\vec{a} \cdot \vec{b} = 0$.

(v) If \vec{a} is any vector, then $\vec{a} \cdot \vec{a} = |\vec{a}|^2$.

(vi) If \vec{a} and \vec{b} are two vectors, then $(m\vec{a}) \cdot \vec{b} = m(\vec{a} \cdot \vec{b}) = \vec{a} \cdot m\vec{b}$, where m is a scalar.

(vii) If \vec{a} and \vec{b} are two vectors, then

$$m\vec{a} \cdot n\vec{b} = mn(\vec{a} \cdot \vec{b}) = (mn\vec{a}) \cdot \vec{b} = \vec{a} \cdot mn\vec{b}, \text{ where } m \text{ and } n \text{ are scalars.}$$

(viii) If \vec{a} and \vec{b} are two vectors, then

$$\vec{a} \cdot (-\vec{b}) = -(\vec{a} \cdot \vec{b}) = (-\vec{a}) \cdot \vec{b}$$

$$(-\vec{a}) \cdot (-\vec{b}) = \vec{a} \cdot \vec{b}$$

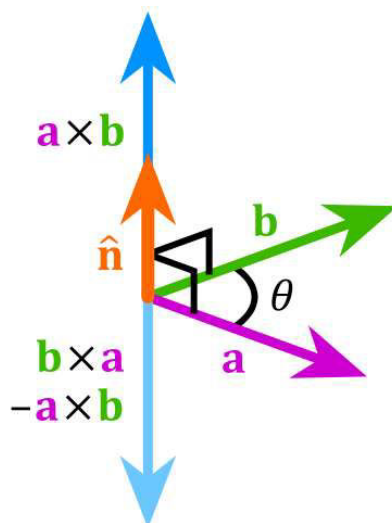
(ix) If \vec{a} and \vec{b} are two vectors, then

$$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b}$$

$$|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}$$

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$$

46. The vector product of two non-zero vectors \vec{a} and \vec{b} denoted by $\vec{a} \times \vec{b}$ is defined as $\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$, where θ is the angle between \vec{a} and \vec{b} , $0 \leq \theta \leq \pi$, and \hat{n} is a unit vector perpendicular to both \vec{a} and \vec{b} . Here, \vec{a}, \vec{b} and \hat{n} form a right handed system.



47. Properties of a vector product:

(i) $\vec{a} \times \vec{b}$ is a vector.

(ii) Vector product is not commutative.

$$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$$

(iii) If \vec{a} and \vec{b} are non-zero vectors, then $\vec{a} \times \vec{b} = 0$ if and only if \vec{a} and \vec{b} are collinear, i.e.

$$\vec{a} \times \vec{b} = 0 \Leftrightarrow \vec{a} \parallel \vec{b}, \text{ here either } \theta = 0 \text{ or } \theta = \pi$$

(iv) If $\theta = \frac{\pi}{2}$, then $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}|$.

(v) If \vec{a} and \vec{b} are two vectors, λ is a scalar, then $\lambda \vec{a} \times \vec{b} = \lambda (\vec{a} \times \vec{b}) = \vec{a} \times \lambda \vec{b}$

(vi) If \vec{a} and \vec{b} are two vectors, and λ and μ are scalars, then

$$\lambda \vec{a} \times \mu \vec{b} = \lambda \mu (\vec{a} \times \vec{b}) = \lambda (\vec{a} \times \mu \vec{b}) = \mu (\lambda \vec{a} \times \vec{b})$$

(vii) Vector product is distributive over addition.

If \vec{a}, \vec{b} and \vec{c} are three vectors, then

$$(i) \vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$$

$$(ii) (\vec{b} + \vec{c}) \times \vec{a} = (\vec{b} \times \vec{a}) + (\vec{c} \times \vec{a})$$

(viii) Vector product is distributive over subtraction.

If \vec{a}, \vec{b} and \vec{c} are three vectors, then

$$(i) \vec{a} \times (\vec{b} - \vec{c}) = (\vec{a} \times \vec{b}) - (\vec{a} \times \vec{c})$$

$$(ii) (\vec{b} - \vec{c}) \times \vec{a} = (\vec{b} \times \vec{a}) - (\vec{c} \times \vec{a})$$

(ix) If we have two vectors \vec{a} and \vec{b} given in component form as,

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \text{ and } \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\text{then } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(x) For unit vectors \hat{i} , \hat{j} and \hat{k} ,

$$\hat{i} \times \hat{i} = 0; \hat{j} \times \hat{j} = 0; \hat{k} \times \hat{k} = 0$$

$$\hat{i} \times \hat{j} = \hat{k}; \hat{j} \times \hat{k} = \hat{i}; \hat{k} \times \hat{i} = \hat{j}$$

$$\hat{j} \times \hat{i} = -\hat{k}; \hat{k} \times \hat{j} = -\hat{i}; \hat{i} \times \hat{k} = -\hat{j}$$

(xi) $\vec{a} \times \vec{a} = \vec{0}$ as $\theta = 0 \therefore \sin\theta = 0$

$$\vec{a} \times (-\vec{a}) = \vec{0} \text{ as } \theta = \pi \therefore \sin\theta = 0$$

$$\vec{a} \perp \vec{b} \Rightarrow \theta = \frac{\pi}{2} \Rightarrow \sin\theta = 1 \Rightarrow \vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \hat{n}$$

(xii) The angle between two non-zero vectors \vec{a} and \vec{b} is given by

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|} \text{ or } \theta = \sin^{-1} \left(\frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|} \right)$$

48. Let \vec{a} and \vec{b} be two vectors. Then the vectors perpendicular to \vec{a} and \vec{b} with magnitude 'k' are given

$$\text{by } \frac{\pm k (\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}.$$

49. The area of a parallelogram is equal to the modulus of the cross product of the vectors representing its adjacent sides.

$$A = |\vec{a} \times \vec{b}|$$

50. The area of a parallelogram with diagonals \vec{c} and \vec{d} is $\frac{1}{2} |\vec{c} \times \vec{d}|$.

51. The area of a triangle is equal to half of the modulus of the cross product of the vectors representing its adjacent sides.

$$A = \frac{1}{2} |\vec{a} \times \vec{b}|$$

52. The area of a ΔABC is $\frac{1}{2} \times |\overrightarrow{AB} \times \overrightarrow{AC}|$ or $\frac{1}{2} \times |\overrightarrow{BC} \times \overrightarrow{BA}|$ or $\frac{1}{2} \times |\overrightarrow{CB} \times \overrightarrow{CA}|$

53. The area of a ΔABC with position vectors of the vertices A, B and C is area of

$$\Delta ABC = \frac{1}{2} |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|.$$

54. The length of the perpendicular from C on AB = $\frac{\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}}{|\vec{a} - \vec{b}|}$.

55. The length of the perpendicular from A on BC = $\frac{\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}}{|\vec{b} - \vec{c}|}$.

56. The length of the perpendicular from B on AC = $\frac{\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}}{|\vec{c} - \vec{a}|}$.

57. The area of a quadrilateral ABCD is $\frac{1}{2} |\overrightarrow{AC} \times \overrightarrow{BD}|$, where, \overrightarrow{AC} and \overrightarrow{BD} are its diagonals.

58. The vector sum of the sides of a triangle taken in order is zero.

Top Formulae

1. Properties of addition of vectors

(i) Vector addition is commutative.

$$\vec{a} + \vec{b} = \vec{b} + \vec{a}$$

(ii) Vector addition is associative.

$$\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$$

(iii) $\vec{0}$ is an additive identity for vector addition.

$$\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$$

2. The position vector of the point C which divides AB internally in the ratio m : n is given

$$\text{by } \overrightarrow{OC} = \frac{m\vec{b} + n\vec{a}}{m + n}.$$

3. The position vector of the point C which divides AB externally in the ratio m : n is given

$$\text{by } \overrightarrow{OC} = \frac{m\vec{b} - n\vec{a}}{m - n}.$$

4. Linear combination of vectors \vec{a} , \vec{b} , and \vec{c} is of the form $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}$, where x, y and z are the scalars.

5. The position vector of the centroid is $\frac{\vec{a} + \vec{b} + \vec{c}}{3}$, where \vec{a}, \vec{b} and \vec{c} are the position vectors of the triangle.

6. Magnitude or length of the vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is $|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

7. Vector addition in component form: Given $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, then
 $\vec{r}_1 + \vec{r}_2 = (x_1 + x_2)\hat{i} + (y_1 + y_2)\hat{j} + (z_1 + z_2)\hat{k}$

8. Difference of vectors: Given $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, then
 $\vec{r}_1 - \vec{r}_2 = (x_1 - x_2)\hat{i} + (y_1 - y_2)\hat{j} + (z_1 - z_2)\hat{k}$

9. Equal vectors: Given $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$, then $\vec{r}_1 = \vec{r}_2$ if and only if
 $x_1 = x_2; y_1 = y_2; z_1 = z_2$

10. Multiplication of $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ with scalar k is given by $k\vec{r} = (kx)\hat{i} + (ky)\hat{j} + (kz)\hat{k}$

11. For any vector, \vec{r} in component form $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then x, y , and z are the direction ratios of \vec{r} and
 $\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}},$ and $\frac{z}{\sqrt{x^2 + y^2 + z^2}}$ are its direction cosines.

12. The direction ratios of the line segment joining points (x_1, y_1, z_1) and (x_2, y_2, z_2) are proportional to
 $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

13. If a vector \vec{r} has direction ratios proportional to a, b and c , then

$$\vec{r} = \frac{|\vec{r}|}{\sqrt{a^2 + b^2 + c^2}} (a\hat{i} + b\hat{j} + c\hat{k}).$$

14. Let \vec{a} and \vec{b} be any two vectors and k and m be two scalars, then

(i) $k\vec{a} + m\vec{a} = (k + m)\vec{a}$

(ii) $k(m\vec{a}) = (km)\vec{a}$

(iii) $k(\vec{a} + \vec{b}) = k\vec{a} + k\vec{b}$

15. Vectors $\vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ are collinear if

$$(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) = k(x_2\hat{i} + y_2\hat{j} + z_2\hat{k})$$

i.e. $x_1 = kx_2$; $y_1 = ky_2$; $z_1 = kz_2$

$$\text{or } \frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2} = k$$

16. The projections of a vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ on the coordinate axes are $l\vec{r}, m\vec{r}, n\vec{r}$, where l, m and n are the direction cosines of the vector \vec{r} .

17. The scalar product of vectors \vec{a} and \vec{b} is $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$, where θ is the angle between vectors \vec{a} and \vec{b} .

18. The scalar product in terms of components: Let \vec{a} and \vec{b} be two vectors such that

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \text{ and } \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}, \text{ then}$$

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

19. The angle between two non-zero vectors \vec{a} and \vec{b} is given by $\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|}$ or $\theta = \cos^{-1} \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \right)$.

20. Let \vec{a} and \vec{b} be two vectors such that $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then the angle between the two non-zero vectors \vec{a} and \vec{b} is given by

$$\theta = \cos^{-1} \left\{ \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \cdot \sqrt{b_1^2 + b_2^2 + b_3^2}} \right\}$$

21. Let \vec{a} and \vec{b} be two vectors. Then

(i) $\vec{a} \cdot \vec{b} = 0$ if and only if \vec{a} is perpendicular to \vec{b} .

(ii) $\vec{a} \cdot \vec{b} > 0$ if and only if θ is acute.

(iii) $\vec{a} \cdot \vec{b} < 0$ if and only if θ is obtuse.

22. Components of a vector along and perpendicular to vector:

Let \vec{a} and \vec{b} be two vectors.

Then the components of \vec{b} along and perpendicular to \vec{a} are

$$\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a} \text{ and } \vec{b} - \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a} \text{ respectively.}$$

23. The projection of \vec{r} on the X, Y and Z axes are x, y and z, respectively, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

24. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then x, y and z are called the components of \vec{r} along X, Y and Z axes, respectively.

25. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is a vector making an angle α, β and γ with the X, Y and Z axes, respectively, then,

$$\vec{r} = |\vec{r}|[(\cos \alpha)\hat{i} + (\cos \beta)\hat{j} + (\cos \gamma)\hat{k}] \text{ and}$$

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

26. For unit vectors \hat{i}, \hat{j} and \hat{k} , $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$ and $\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

27. The unit vector in the direction of vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is $\frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$

28.

Projection of a vector \vec{a} on other vector \vec{b} is given by:

$$\vec{a} \cdot \hat{b} = \vec{a} \cdot \left(\frac{\vec{b}}{|\vec{b}|} \right) = \frac{1}{|\vec{b}|} (\vec{a} \cdot \vec{b})$$

29. Cauchy–Schwarz Inequality: $|\vec{a} \cdot \vec{b}| \leq |\vec{a}| \cdot |\vec{b}|$

30. Triangle Inequality: $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$

31. The vector product of vectors a and b is $\vec{a} \times \vec{b} = |\vec{a}| |\vec{b}| \sin \theta \hat{n}$.

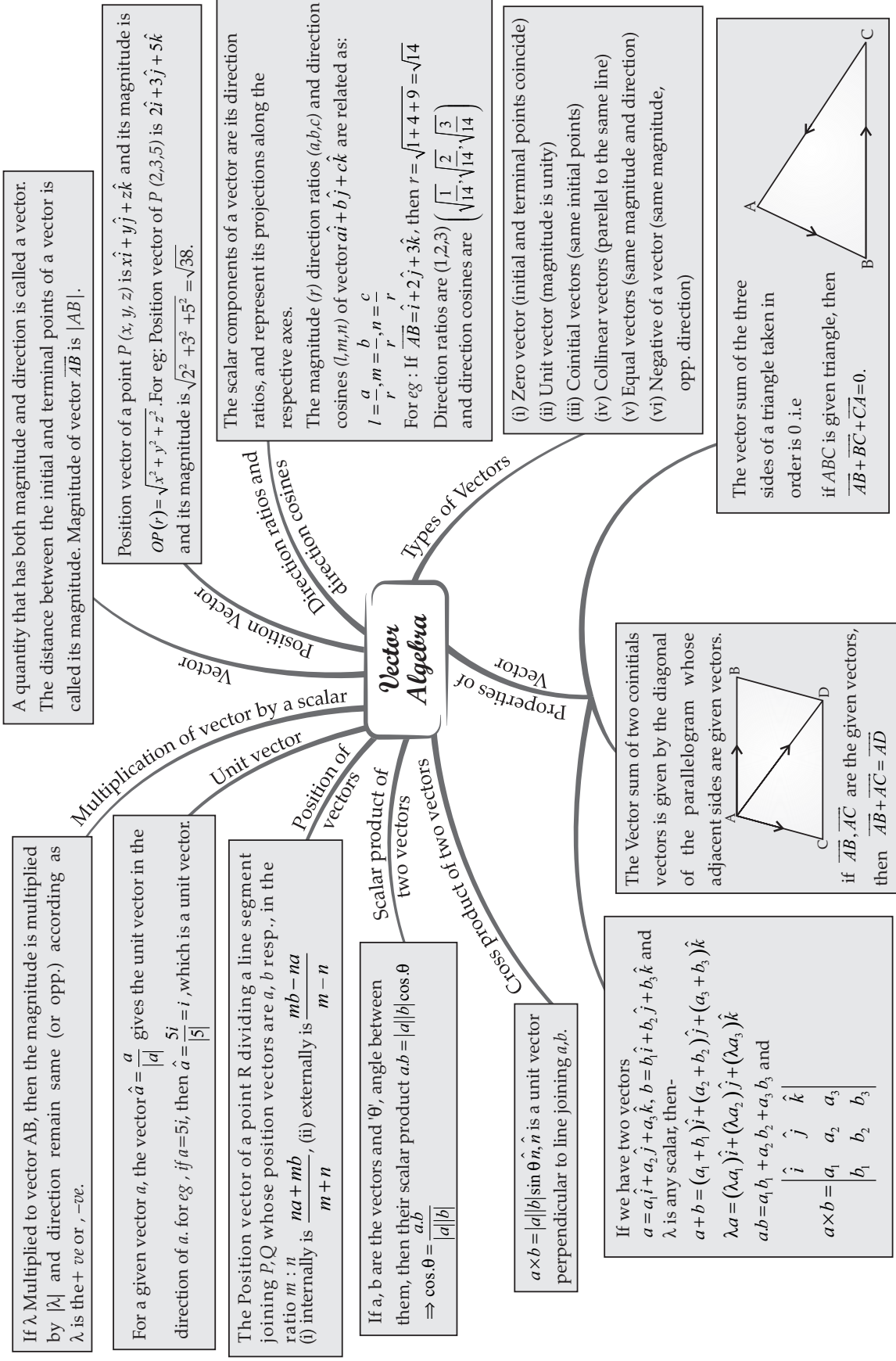
32. Geometrical meaning of vector product:

Let \vec{a} and \vec{b} be two non-zero, non-parallel vectors.

Then the cross-product $\vec{a} \times \vec{b}$ is a vector with magnitude equal to the area of the parallelogram having \vec{a} and \vec{b} as its adjacent sides and direction \hat{n} is perpendicular to the plane of \vec{a} and \vec{b} .

MIND MAP : LEARNING MADE SIMPLE

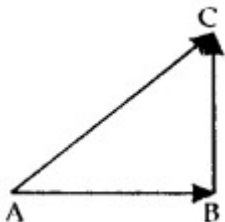
CHAPTER - 10



Important Questions

Multiple Choice questions-

1. In $\triangle ABC$, which of the following is not true?



(a) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$

(b) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$

(c) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0}$

(d) $\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$

2. If \vec{a} and \vec{b} are two collinear vectors, then which of the following are incorrect:

(a) $\vec{b} = \lambda \vec{a}$ for some scalar λ .

(b) $\vec{a} = \pm \vec{b}$

(c) the respective components of \vec{a} and \vec{b} are proportional

(d) both the vectors \vec{a} and \vec{b} have the same direction, but different magnitudes.

3. If \vec{a} is a non-zero vector of magnitude 'a' and λ a non-zero scalar, then $\lambda \vec{a}$ is unit vector if:

(a) $\lambda = 1$

(b) $\lambda = -1$

(c) $a = |\lambda|$

(d) $a = \frac{1}{|\lambda|}$

4. Let λ be any non-zero scalar. Then for what possible values of x , y and z given below, the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $x\hat{i} - y\hat{j} + z\hat{k}$ are perpendicular:

(a) $x = 2\lambda$, $y = \lambda$, $z = \lambda$

(b) $x = \lambda, y = 2\lambda, z = -\lambda$

(c) $x = -\lambda, y = 2\lambda, z = \lambda$

(d) $x = -\lambda, y = -2\lambda, z = \lambda$.

5. Let the vectors \vec{a} and \vec{b} be such that $|\vec{a}| = 3$ and $|\vec{b}| = \frac{\sqrt{2}}{3}$, then $\vec{a} \times \vec{b}$ is a unit vector if the angle between \vec{a} and \vec{b} is:

(a) $\frac{\pi}{6}$

(b) $\frac{\pi}{4}$

(c) $\frac{\pi}{3}$

(d) $\frac{\pi}{2}$

6. Area of a rectangle having vertices

A $(-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}),$

B $(\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}),$

C $(\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}),$

D $(-\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k})$ is

(a) $\frac{1}{2}$ square unit

(b) 1 square unit

(c) 2 square units

(d) 4 square units.

7. If θ is the angle between two vectors \vec{a}, \vec{b} , then $\vec{a} \cdot \vec{b} \geq 0$ only when

(a) $0 < \theta < \frac{\pi}{2}$

(b) $0 \leq \theta \leq \frac{\pi}{2}$

(c) $0 < \theta < \pi$

(d) $0 \leq \theta \leq \pi$

8. Let \vec{a} and \vec{b} be two unit vectors and θ is the angle between them. Then $\vec{a} + \vec{b}$ is a unit vector if:

(a) $\theta = \frac{\pi}{4}$

(b) $\theta = \frac{\pi}{3}$

(c) $\theta = \frac{\pi}{2}$

(d) $\theta = \frac{2\pi}{3}$

9. If $\{\hat{i}, \hat{j}, \hat{k}\}$ are the usual three perpendicular unit vectors, then the value of:

$\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j})$ is

(a) 0

(b) -1

(c) 1

(d) 3

10. If θ is the angle between two vectors \vec{a} and \vec{b} , then $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$ when θ is equal to:

(a) 0

(b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$

(d) π

Very Short Questions:

1. Classify the following measures as scalar and vector quantities:

(i) 40°

(ii) 50 watt

(iii) 10gm/cm^3

(iv) 20 m/sec towards north

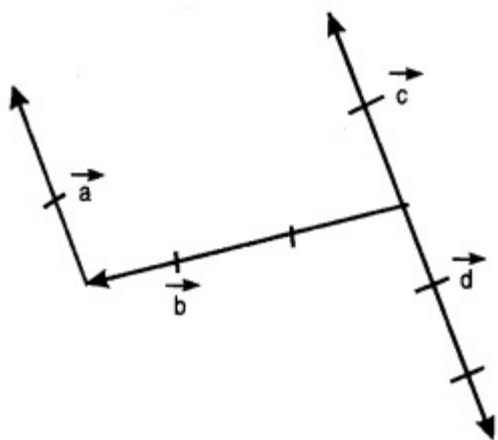
(v) 5 seconds. (N.C.E.R.T.)

2. In the figure, which of the vectors are:

(i) Collinear

(ii) Equal

(iii) Co-initial. (N.C.E.R.T.)



3. Find the sum of the vectors:

$$\vec{a} = \hat{i} - 2\hat{j} + \hat{k}, \vec{a} = -2\hat{i} + 4\hat{j} + 5\hat{k} \text{ and } \vec{c} = \hat{i} - 6\hat{j} - 7\hat{k} \text{ (C.B.S.E. 2012)}$$

4. Find the vector joining the points P (2,3,0) and Q (-1, - 2, - 4) directed from P to Q. (N.C.E.R.T.)

5. If $\vec{a} = x\hat{i} + 2\hat{j} - z\hat{k}$ and $\vec{b} = 3\hat{i} - y\hat{j} + \hat{k}$ are two equal vectors, then write the value of $x + y + z$. (C.B.S.E. 2013)

6. Find the unit vector in the direction of the sum of the vectors:

$$\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k} \text{ and } \vec{b} = -\hat{i} + \hat{j} + 3\hat{k} \text{ (N.C.E.R.T.)}$$

7. Find the value of 'p' for which the vectors: $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} - 2p\hat{j} + 3\hat{k}$ are parallel. (A.I.C.B.S.E. 2014)

8. If \vec{a} and \vec{b} are perpendicular vectors, $|\vec{a} + \vec{b}| = 13$ and $|\vec{a}| = 5$, find the value of $|\vec{b}|$ (A.I.C.B.S.E. 2014)

9. Find the magnitude of each of the two vectors \vec{a} and \vec{b} , having the same magnitude such

that the angle between them is 60° and their scalar product is $\frac{9}{2}$ (C.B.S.E. 2018)

10. Find the area of the parallelogram whose diagonals are represented by the vectors: $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$ (C.B.S.E. Sample Paper 2018-19)

Short Questions:

1. If θ is the angle between two vectors:

$$\hat{i} - 2\hat{j} + 3\hat{k} \text{ and } 3\hat{i} - 2\hat{j} + \hat{k}, \text{ find } \sin\theta. \text{ (C.B.S.E. 2018)}$$

2. X and Y are two points with position vectors $\vec{3a} + \vec{b}$ and $\vec{a} - \vec{3b}$ respectively. Write the position vector of a point Z which divides the line segment XY in the ratio 2:1 externally. (C.B.S.E. Outside Delhi 2019)

3. Find the unit vector perpendicular to both \vec{a} and \vec{b} , where:

$$\vec{a} = 4\hat{i} - \hat{j} + 8\hat{k} \text{ and } \vec{b} = -\hat{j} + \hat{k}$$

4. If $\vec{a} = 2\hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to \vec{c} , then find the value of λ . (C.B.S.E. 2019 C)

5. Let $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$ be two vectors. Show that the vectors $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are perpendicular to each other. (C.B.S.E. Outside Delhi 2019)

6. If the sum of two-unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$. (C.B.S.E. 2019)

7. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 3$, $|\vec{b}| = 5$ and $|\vec{c}| = 7$, then find the value of $\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$ (C.B.S.E. Sample Paper 2019-20)

8. Find $|\vec{a} - \vec{b}|$, if two vectors a and b are such that $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 4$. (N.C.E.R.T.)

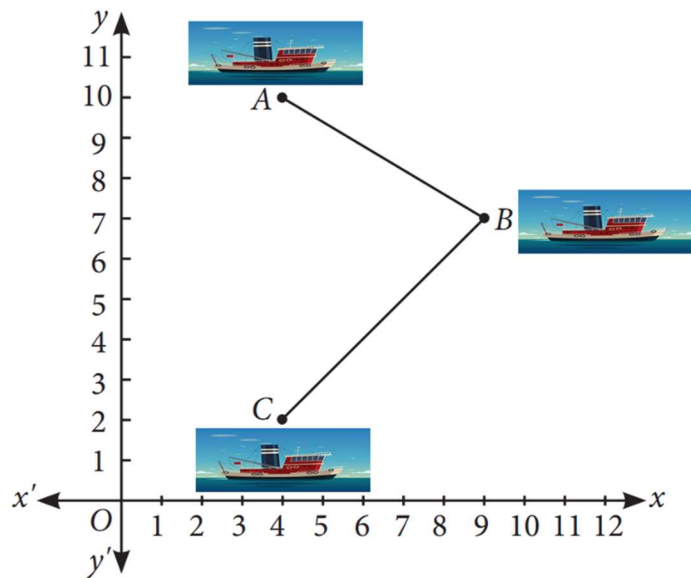
Long Questions:

1. Let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$. Find a vector \vec{d} which is perpendicular to both \vec{c} and \vec{b} and $\vec{d} \cdot \vec{a} = 21$. (C.B.S.E. 2018)
2. If $\vec{p} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{q} = \hat{i} - 2\hat{j} + \hat{k}$, find a vector of magnitude $5\sqrt{3}$ units perpendicular to the vector \vec{q} and coplanar with vector \vec{p} and \vec{q} . (C.B.S.E. 2018)

3. If $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 5\hat{j}$, $3\hat{i} + 2\hat{j} - 3\hat{k}$ and $\hat{i} - 6\hat{j} - \hat{k}$ respectively are the position vectors of points A, B, C and D, then find the angle between the straight lines AB and CD. Find
4. If $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 3$, $|\vec{b}| = 5$, and $|\vec{c}| = 7$, find the angle between \vec{a} and \vec{b} . (C.B.S.E. 2014)

Case Study Questions:

1. A barge is pulled into harbour by two tug boats as shown in the figure.



Based on the above information, answer the following questions.

- i. Position vector of A is:

- a. $4\hat{i} + 2\hat{j}$
- b. $4\hat{i} + 10\hat{j}$
- c. $4\hat{i} - 10\hat{j}$
- d. $4\hat{i} - 2\hat{j}$

ii. Position vector of B is:

a. $4\hat{i} + 4\hat{j}$

b. $6\hat{i} + 6\hat{j}$

c. $9\hat{i} + 7\hat{j}$

d. $3\hat{i} + 3\hat{j}$

iii. Find the vector \overrightarrow{AC} in terms of \hat{i}, \hat{j} .

a. $8\hat{j}$

b. $-8\hat{j}$

c. $8\hat{i}$

d. None of these

iv. If $\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}$, then its unit vector is:

a. $\frac{\hat{i}}{\sqrt{14}} + \frac{2\hat{j}}{\sqrt{14}} + \frac{3\hat{k}}{\sqrt{14}}$

b. $\frac{3\hat{i}}{\sqrt{14}} + \frac{2\hat{j}}{\sqrt{14}} + \frac{\hat{k}}{\sqrt{14}}$

c. $\frac{2\hat{i}}{\sqrt{14}} + \frac{3\hat{j}}{\sqrt{14}} + \frac{\hat{k}}{\sqrt{14}}$

d. None of these

v. If $\vec{A} = 4\hat{i} + 3\hat{j}$ and $\vec{B} = 3\hat{i} + 4\hat{j}$, then $|\vec{A}| + |\vec{B}| =$

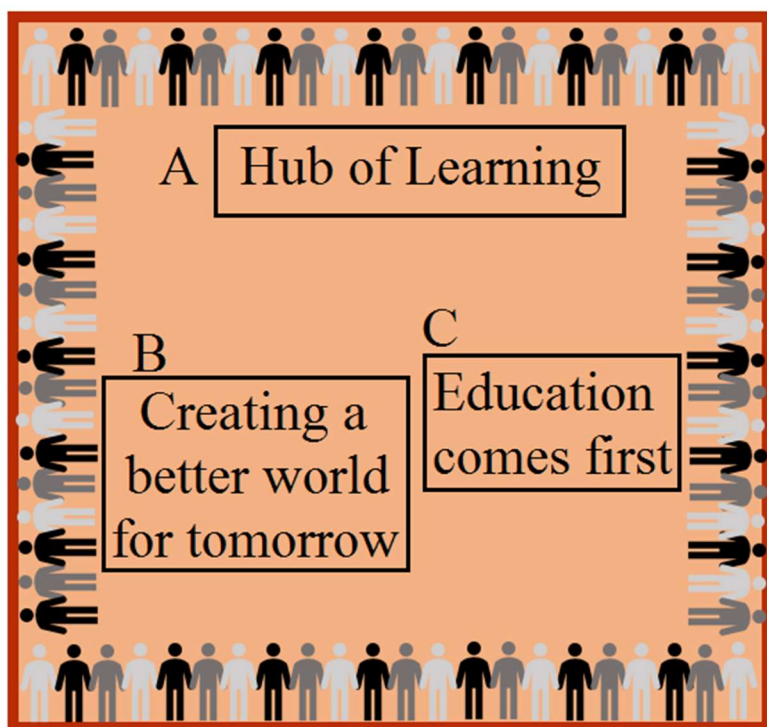
a. 12

b. 13

c. 14

d. 10

2. Three slogans on chart papers are to be placed on a school bulletin board at the points A, B and C displaying A (Hub of Learning), B (Creating a better world for tomorrow) and C (Education comes first). The coordinates of these points are (1, 4, 2), (3, -3, -2) and (-2, 2, 6) respectively.



Based on the above information, answer the following questions.

i. Let \vec{a} , \vec{b} and \vec{c} be the position vectors of points A, B

and C respectively, then $\vec{a} + \vec{b} + \vec{c}$ is equal to:

- a. $2\hat{i} + 3\hat{j} + 6\hat{k}$
- b. $2\hat{i} - 3\hat{j} - 6\hat{k}$
- c. $2\hat{i} + 8\hat{j} + 3\hat{k}$
- d. $2(7\hat{i} + 8\hat{j} + 3\hat{k})$

ii. Which of the following is not true?

- a. $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$
- b. $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{AC} = \vec{0}$
- c. $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0}$
- d. $\overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$

iii. Area of $\triangle ABC$ is:

- a. 19 sq. units
- b. $\sqrt{1937}$ sq. units
- c. $\frac{1}{2}\sqrt{1937}$ sq. units
- d. $\sqrt{1837}$ sq. units

iv. Suppose, if the given slogans are to be placed on a straight line,

then the value of $|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$ will be equal to:

- a. -1
- b. -2
- c. 2
- d. 0

v. If $\vec{a} = 2\hat{i} + 3\hat{j} + 6\hat{k}$, then unit vector in the direction of vector \vec{a} is:

- a. $\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}$
- b. $\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$
- c. $\frac{3}{7}\hat{i} + \frac{2}{7}\hat{j} + \frac{6}{7}\hat{k}$
- d. None of these

Answer Key-

Multiple Choice questions-

1. Answer: (c) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0}$
2. Answer: (d) both the vectors \vec{a} and \vec{b} have the same direction, but different magnitudes.
3. Answer: (d) $a = \frac{1}{|\lambda|}$
4. Answer: (c) $x = -\lambda, y = 2\lambda, z = \lambda$
5. Answer: (b) $\frac{\pi}{4}$
6. Answer: (c) 2 square units

7. Answer: (b) $0 \leq \theta \leq \frac{\pi}{2}$

8. Answer: (d) $\theta = \frac{2\pi}{3}$

9. Answer: (d) 3

10. Answer: (b) $\frac{\pi}{4}$

Very Short Answer:

1. Solution:

(i) Angle-scalar

(ii) Power-scalar

(iii) Density-scalar

(iv) Velocity-vector

(v) Time-scalar.

2. Solution:

(i) \vec{a} , \vec{c} and \vec{d} are collinear vectors.

(ii) \vec{a} and \vec{c} are equal vectors.

(iii) \vec{b} , \vec{c} and \vec{d} are co-initial vectors.

3. Solution:

Sum of the vectors = $\hat{a} + \hat{b} + \hat{c}$

$$= (\hat{i} - 2\hat{j} + \hat{k}) + (-2\hat{i} + 4\hat{j} + 5\hat{k}) + (\hat{i} - 6\hat{j} - 7\hat{k})$$

$$= (\hat{i} - 2\hat{i} + \hat{i}) + (-2\hat{j} + 4\hat{j} - 6\hat{j}) + (\hat{k} + 5\hat{k} - 7\hat{k})$$

$$= -4\hat{j} - \hat{k}.$$

4. Solution:

Since the vector is directed from P to Q,

\therefore P is the initial point and Q is the terminal point.

$$\therefore \text{Reqd. vector} = \overrightarrow{PQ}$$

$$= (-\hat{i} - 2\hat{j} - 4\hat{k}) - (2\hat{i} + 3\hat{j} + 0\hat{k})$$

$$= (-1 - 2)\hat{i} + (-2 - 3)\hat{j} + (-4 - 0)\hat{k}$$

$$= -3\hat{i} - 5\hat{j} - 4\hat{k}.$$

5. Solution:

Here

$$\vec{a} = \vec{b} \Rightarrow x\hat{i} + 2\hat{j} - z\hat{k} = 3\hat{i} - y\hat{j} + \hat{k}$$

Comparing, $A = 3, 2 = -y$ i.e. $y = -2$, $-z = 1$ i.e. $z = -1$.

Hence, $x + y + z = 3 - 2 - 1 = 0$.

6. Solution:

$$\text{We have : } \vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$$

$$\text{and } \vec{b} = -\hat{i} + \hat{j} + 3\hat{k}.$$

$$\therefore \vec{c} = \vec{a} + \vec{b}$$

$$= (2\hat{i} - \hat{j} + 2\hat{k}) + (-\hat{i} + \hat{j} + 3\hat{k})$$

$$= \hat{i} + 0\hat{j} + 5\hat{k}.$$

$$\therefore |\vec{c}| = \sqrt{1^2 + 0^2 + 5^2}$$

$$= \sqrt{1+0+25} = \sqrt{26}.$$

$$\therefore \text{Reqd. unit vector} = \hat{c} = \frac{\vec{c}}{|\vec{c}|}$$

$$= \frac{\hat{i} + 0\hat{j} + 5\hat{k}}{\sqrt{26}} = \frac{\hat{i} + 5\hat{k}}{\sqrt{26}}.$$

7. Solution:

The given vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} - 2\hat{j} + 3\hat{k}$ are parallel

$$\text{If } \frac{3}{1} = \frac{2}{-2p} = \frac{9}{3} \text{ if } 3 = \frac{1}{-p} = 3$$

$$\text{if } p = -\frac{1}{3}$$

8. Solution:

$$\text{We have : } |\vec{a} + \vec{b}| = 13.$$

$$\text{Squaring, } (\vec{a} + \vec{b})^2 = 169$$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b} = 169$$

$$\Rightarrow (5)^2 + |\vec{b}|^2 + 2(0) = 169$$

$$[\because \vec{a} \text{ and } \vec{b} \text{ are perpendicular} \Rightarrow \vec{a} \cdot \vec{b} = 0]$$

$$\Rightarrow |\vec{b}|^2 = 169 - 25 = 144.$$

$$\text{Hence, } |\vec{b}| = 12.$$

9. Solution:

$$\text{By the question, } |\vec{a}| = |\vec{b}| \dots (1)$$

$$\text{Now } \vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$$

$$\Rightarrow \frac{9}{2} = |\vec{a}||\vec{a}| \cos 60^\circ \text{ [Using (1)]}$$

$$\Rightarrow \frac{9}{2} = |\vec{a}|^2 \left(\frac{1}{2}\right)$$

$$\Rightarrow |\vec{a}|^2 = 9.$$

$$\text{Hence, } |\vec{a}| = |\vec{b}| = 3.$$

10. Solution:

$$\text{We have: } \vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\text{and } \vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}.$$

$$\therefore \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 4 \\ 2 & -1 & 2 \end{vmatrix}$$

$$\begin{aligned}
 &= \hat{i}(-6+4) - \hat{j}(4-8) + \hat{k}(-2+6) \\
 &= -2\hat{i} + 4\hat{j} + 4\hat{k}. \\
 \therefore |\vec{a} \times \vec{b}| &= \sqrt{4+16+16} = \sqrt{36} = 6. \\
 \therefore \text{Area of the parallelogram} &= \frac{1}{2} |\vec{a} \times \vec{b}| \\
 &= \frac{1}{2} (6) = 3 \text{ sq. units.}
 \end{aligned}$$

Short Answer:

1. Solution:

$$\begin{aligned}
 \text{We know that } \sin \theta &= \frac{|\vec{a} \times \vec{b}|}{|\vec{a}| |\vec{b}|} \\
 \Rightarrow \sin \theta &= \frac{(\hat{i} - 2\hat{j} + 3\hat{k}) \times (3\hat{i} - 2\hat{j} + \hat{k})}{|\hat{i} - 2\hat{j} + 3\hat{k}| |3\hat{i} - 2\hat{j} + \hat{k}|} \quad \dots(1)
 \end{aligned}$$

$$\text{Now } (\hat{i} - 2\hat{j} + 3\hat{k}) \times (3\hat{i} - 2\hat{j} + \hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 3 & -2 & 1 \end{vmatrix}$$

$$= \hat{i}(-2+6) - \hat{j}(1-9) + \hat{k}(-2+6)$$

$$= 4\hat{i} + 8\hat{j} + 4\hat{k}.$$

$$\therefore |4\hat{i} + 8\hat{j} + 4\hat{k}| = \sqrt{16+64+16}$$

$$= \sqrt{96} = 4\sqrt{6}$$

$$\text{and } |\hat{i} - 2\hat{j} + 3\hat{k}| = \sqrt{1+4+9} = \sqrt{14};$$

$$|3\hat{i} - 2\hat{j} + \hat{k}| = \sqrt{9+4+1} = \sqrt{14}.$$

$$\therefore \text{From (1), } \sin \theta = \frac{4\sqrt{6}}{\sqrt{14}\sqrt{14}} = \frac{4\sqrt{6}}{14}.$$

$$\text{Hence, } \sin \theta = \frac{2\sqrt{6}}{7}.$$

2. Solution:

Position vector of

$$A = \frac{2(\vec{a}-3\vec{b})-(3\vec{a}+\vec{b})}{2-1} = -\vec{a} - 7\vec{b}$$

3. Solution:

We have : $\vec{a} = 4\hat{i} - \hat{j} + 8\hat{k}$, $\vec{b} = -\hat{j} + \hat{k}$.

$$\begin{aligned}\therefore \vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 8 \\ 0 & -1 & 1 \end{vmatrix} \\ &= \hat{i}(-1+8) - \hat{j}(4-0) + \hat{k}(-4+0) \\ &= 7\hat{i} - 4\hat{j} - 4\hat{k}. \\ \therefore |\vec{a} \times \vec{b}| &= \sqrt{(7)^2 + (-4)^2 + (-4)^2} \\ &= \sqrt{49+16+16} = \sqrt{81} = 9.\end{aligned}$$

Hence, the unit vector perpendicular to both \vec{a} and \vec{b}

$$= \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{7\hat{i} - 4\hat{j} - 4\hat{k}}{9} = \frac{7}{9}\hat{i} - \frac{4}{9}\hat{j} - \frac{4}{9}\hat{k}.$$

4. Solution:

We have:

$$\begin{aligned}a = \vec{a} &= 2\hat{i} + 2\hat{j} + \hat{k} \text{ and } \vec{b} = -\hat{i} + 2\hat{j} + \hat{k} \\ \therefore \vec{a} + \lambda\vec{b} &= (2\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + 2\hat{j} + \hat{k}) \\ &= (2 - \lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (3 + \lambda)\hat{k}.\end{aligned}$$

Now, $(\vec{a} + \lambda\vec{b})$ is perpendicular to \vec{c} ,

$$\therefore (\vec{a} + \lambda\vec{b}) \cdot \vec{c} = 0$$

$$\Rightarrow ((2 - \lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (3 + \lambda)\hat{k}) \cdot (3\hat{i} + \hat{j}) = 0$$

$$\Rightarrow (2 - \lambda)(3) + (2 + 2\lambda)(1) + (3 + \lambda)(0) = 0$$

$$\Rightarrow 6 - 3\lambda + 2 + 2\lambda = 0$$

$$\Rightarrow -\lambda, + 8 = 0.$$

$$\text{Hence, } \lambda, = 8.$$

5. Solution:

$$\text{Here, } \vec{a} + \vec{b} = (\hat{i} + 2\hat{j} - 3\hat{k}) + (3\hat{i} - \hat{j} + 2\hat{k})$$

$$= 4\hat{i} + \hat{j} - \hat{k}$$

$$\text{and } \vec{a} - \vec{b} = (\hat{i} + 2\hat{j} - 3\hat{k}) - (3\hat{i} - \hat{j} + 2\hat{k})$$

$$= -2\hat{i} + 3\hat{j} - 5\hat{k}.$$

Now,

$$\vec{a} + \vec{b} \cdot \vec{a} - \vec{b} = (4\hat{i} + \hat{j} - \hat{k}) \cdot (-2\hat{i} + 3\hat{j} - 5\hat{k})$$

$$= (4)(-2) + (1)(3) + (-1)(-5)$$

$$= -8 + 3 + 5 = 0.$$

$$\text{Hence } \vec{a} + \vec{b} \text{ is perpendicular to } \vec{a} - \vec{b}.$$

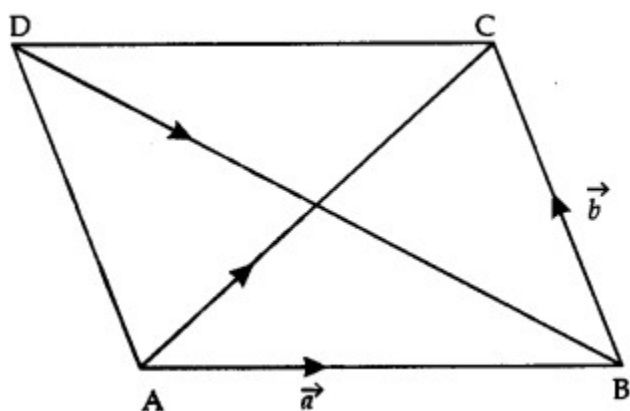
6. Solution:

$$\text{We have : } |\vec{a}| = |\vec{b}| = 1, |\vec{a} + \vec{b}| = 1.$$

$$\text{Let } \vec{AB} = \vec{a}, \vec{BC} = \vec{b}.$$

$$\text{Then, } \vec{AC} = \vec{AB} + \vec{BC} = \vec{a} + \vec{b}$$

$$\begin{aligned} \text{and } \vec{DB} &= \vec{DA} + \vec{AB} \\ &= -\vec{AD} + \vec{AB} = \vec{AB} - \vec{AD} \\ &= \vec{a} - \vec{b}. \end{aligned}$$



By the question,

$$|\vec{AB}| = |\vec{BC}| = |\vec{AC}| = 1$$

$\Rightarrow \Delta ABC$ is equilateral, each of its angles being 60°

$\Rightarrow \angle DAB = 2 \times 60^\circ = 120^\circ$ and $\angle ADB = 30^\circ$.

By Sine Formula,

$$\frac{DB}{\sin \angle DAB} = \frac{AB}{\sin \angle ADB}$$

$$\Rightarrow \frac{|\overline{DB}|}{\sin 120^\circ} = \frac{|\overline{AB}|}{\sin 30^\circ}$$

$$\begin{aligned} \Rightarrow |\overline{DB}| &= \frac{\sin 120^\circ}{\sin 30^\circ} |\overline{AB}| \\ &= \frac{\sqrt{3}/2}{1/2} \times 1 = \sqrt{3}. \end{aligned}$$

$$\text{Hence, } |\vec{a} - \vec{b}| = \sqrt{3}.$$

7. Solution:

Here, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$$\Rightarrow (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) = 0$$

$$\Rightarrow \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} + \vec{c} \cdot \vec{c} = 0$$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$

$$\Rightarrow 3^2 + 5^2 + 7^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$

$$\Rightarrow 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = -(9 + 25 + 49).$$

$$\text{Hence, } \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -\frac{83}{2}.$$

8. Solution:

Here, $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$$\Rightarrow (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) = 0$$

$$\Rightarrow \vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} + \vec{c} \cdot \vec{c} = 0$$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$

$$\Rightarrow 3^2 + 5^2 + 7^2 + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0$$

$$\Rightarrow 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = -(9 + 25 + 49).$$

$$\text{Hence, } \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -\frac{83}{2}.$$

Long Answer:

1. Solution:

$$\text{We have: } \vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$$

$$\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k} \text{ and}$$

$$\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$$

$$\text{Let } \vec{d} = x\hat{i} + y\hat{j} + z\hat{k}$$

since \vec{d} is perpendicular to both \vec{c} and \vec{b}

$$\vec{d} \cdot \vec{c} = 0 \text{ and } \vec{d} \cdot \vec{b} = 0$$

$$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (3\hat{i} + \hat{j} - \hat{k}) = 0$$

$$\text{and } (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} - 4\hat{j} + 5\hat{k}) = 0$$

$$\Rightarrow 3x + y - z = 0 \dots(1)$$

$$\text{and } x - 4y + 5z = 0 \dots(2)$$

$$\text{Also, } \vec{d} \cdot \vec{a} = 21$$

$$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (4\hat{i} + 5\hat{j} - \hat{k}) = 21$$

$$\Rightarrow 4x + 5y - z = 21 \dots(3)$$

Multiplying (1) by 5,

$$15x + 5y - 5z = 0 \dots(4)$$

Adding (2) and (4),

$$16x + y = 0 \dots(5)$$

Subtracting (1) from (3),

$$x + 4y = 21 \dots(6)$$

From (5),

$$y = -16x \dots(7)$$

Putting in (6),

$$x - 64x = 21$$

$$-63x = 21$$

$$\text{Putting in (7), } y = -16 \left(-\frac{1}{3}\right) = \frac{16}{3}$$

$$\text{Putting in (1), } 3 \left(-\frac{1}{3}\right) + \frac{16}{3} - z = 0$$

$$z = 13/3$$

$$\text{Hence } \vec{d} = -\frac{1}{3}\hat{i} + \frac{16}{3}\hat{j} + \frac{13}{3}\hat{k}$$

2. Solution:

Let $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ be the vector.

Since $\vec{r} \perp \vec{q}$

$$(1) (a) + (-2) (b) + 1 (c) = 0$$

$$\Rightarrow a - 2b + c = 0$$

Again, \vec{p} , \vec{q} and \vec{r} are coplanar,

$$\therefore [\vec{p} \ \vec{q} \ \vec{r}] = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ a & b & c \end{vmatrix} = 0$$

$$\Rightarrow (1) (-2c - b) - (1) (c - a) + (1) (b + 2a) = 0$$

$$\Rightarrow -2c - b - c + a + b + 2a = 0$$

$$\Rightarrow 3a - 3c = 0$$

$$\Rightarrow a - c = 0$$

Solving (1) and (2),

$$\frac{a}{2-0} = \frac{b}{1+1} = \frac{c}{0+2}$$

$$\Rightarrow \frac{a}{2} = \frac{b}{2} = \frac{c}{2}$$

$$\Rightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{1}.$$

$$\therefore \vec{r} = 1\hat{i} + 1\hat{j} + 1\hat{k}.$$

$$\therefore |\vec{r}| = \sqrt{3}.$$

$$\therefore \text{Unit vector } \hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}.$$

Hence, the required vector = $5\sqrt{3} \hat{r}$

$$= 5\sqrt{3} \left(\frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}} \right) = 5(\hat{i} + \hat{j} + \hat{k}).$$

3. Solution:

Note: If ' θ ' is the angle between AB and CD,

then θ is also the angle between \overrightarrow{AB} and \overrightarrow{CD} .

Now \overrightarrow{AB} = Position vector of B - Position vector of A

$$= (2\hat{i} + 5\hat{j}) - (\hat{i} + \hat{j} + \hat{k}) = \hat{i} + 4\hat{j} - \hat{k}.$$

$$\therefore |\overrightarrow{AB}| = \sqrt{(1)^2 + (4)^2 + (-1)^2} = 3\sqrt{2}.$$

$$\text{Similarly, } \overrightarrow{CD} = -2\hat{i} - 8\hat{j} + 2\hat{k}$$

$$\text{and } |\overrightarrow{CD}| = 6\sqrt{2}.$$

$$\text{Thus } \cos \theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{CD}}{|\overrightarrow{AB}| |\overrightarrow{CD}|}$$

$$= \frac{1(-2) + 4(-8) + (-1)(2)}{(3\sqrt{2})(6\sqrt{2})} = \frac{-36}{36} = -1.$$

Since $0 \leq \theta \leq \pi$, it follows that $\theta = \pi$. This shows that \overrightarrow{AB} and \overrightarrow{CD} are collinear.

Alternatively, $\overrightarrow{AB} = -\frac{1}{2}\overrightarrow{CD}$ which implies

that \overrightarrow{AB} and \overrightarrow{CD} are collinear vectors.

4. Solution:

$$\text{Since } \vec{a} + \vec{b} + \vec{c} = \vec{0},$$

$$\therefore \vec{a} + \vec{b} = -\vec{c}.$$

$$\text{Squaring, } (\vec{a} + \vec{b})^2 = \vec{c}^2$$

$$\Rightarrow \vec{a}^2 + \vec{b}^2 + 2\vec{a} \cdot \vec{b} = \vec{c}^2$$

$$\Rightarrow \left| \vec{a} \right|^2 + \left| \vec{b} \right|^2 + 2 \left| \vec{a} \right| \left| \vec{b} \right| \cos \theta = \left| \vec{c} \right|^2,$$

where 'θ' is the angle between a and b

$$\Rightarrow (3)^2 + (5)^2 + 2(3)(5) \cos \theta = (7)^2$$

$$\Rightarrow 9 + 25 + 30 \cos \theta = 49$$

$$\Rightarrow 30 \cos \theta = 49 - 34 \Rightarrow \cos \theta = \frac{1}{2}$$

$$\Rightarrow \theta = 60^\circ.$$

Hence, the angle between \vec{a} and \vec{b} is 60° .

Case Study Answers:

1. Answer :

$$\text{i. (b) } 4\hat{i} + 10\hat{j}$$

Solution:

Here, (4, 10) are the coordinates of A.

$$\therefore \text{P.V of A} = 4\hat{i} + 10\hat{j}$$

ii. (c) $9\hat{i} + 7\hat{j}$

Solution:

Here, (9, 7) are the coordinates of B.

$$\therefore \text{P.V of B} = 9\hat{i} + 7\hat{j}$$

iii. (b) $-8\hat{j}$

Solution:

Here, P.V. of $A = 4\hat{i} + 10\hat{j}$ and P.V. of

$$C = 4\hat{i} + 2\hat{j}$$

$$\therefore \overline{AC} = (4 - 4)\hat{i} + (2 - 10)\hat{j} = -8\hat{j}$$

iv. (a) $\frac{\hat{i}}{\sqrt{14}} + \frac{2\hat{j}}{\sqrt{14}} + \frac{3\hat{k}}{\sqrt{14}}$

Solution:

Here, $\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}$

$$\therefore |\vec{A}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$\therefore \vec{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}}$$

$$= \frac{1}{\sqrt{14}}\hat{i} + \frac{2}{\sqrt{14}}\hat{j} + \frac{3}{\sqrt{14}}\hat{k}$$

v. (d) 10

Solution:

We have, $\vec{A} = 4\hat{i} + 3\hat{j}$ and $\vec{B} = 3\hat{i} + 4\hat{j}$

$$\therefore |\vec{A}| = \sqrt{4^2 + 3^2} = \sqrt{16 + 9} = \sqrt{25} = 5$$

$$\text{And } |\vec{B}| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\text{Thus, } |\vec{A}| + |\vec{B}| = 5 + 5 = 10.$$

2. Answer :

i. (a) $2\hat{i} + 3\hat{j} + 6\hat{k}$

Solution:

$$\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}, \vec{b} = 3\hat{i} - 3\hat{j} - 2\hat{k}$$

$$\text{And } \vec{c} = 2\hat{i} + 2\hat{j} + 6\hat{k}$$

$$\therefore \vec{a} + \vec{b} + \vec{c} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

ii. (c) $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0}$

Solution:

Using triangle law of addition in $\triangle ABC$,

we get $\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0}$ which can be rewritten as,

$$\overrightarrow{AB} + \overrightarrow{BC} - \overrightarrow{CA} = \vec{0} \text{ or } \overrightarrow{AB} - \overrightarrow{CB} + \overrightarrow{CA} = \vec{0}$$

iii. (c) $\frac{1}{2}\sqrt{1937}\text{sq. units}$

Solution:

We have, A(1, 4, 2), B(3, -3, -2) and C(-2, 2, 6)

$$\text{Now, } \overrightarrow{AB} = \vec{b} - \vec{a} = 2\hat{i} - 7\hat{j} - 4\hat{k}$$

$$\text{And } \overrightarrow{AC} = \vec{c} - \vec{a} = -3\hat{i} - 2\hat{j} + 4\hat{k}$$

$$\therefore \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -7 & -4 \\ -3 & -2 & 4 \end{vmatrix}$$

$$= \hat{i}(-28 - 8) - \hat{j}(8 - 12) + \hat{k}(-4 - 21)$$

$$= -36\hat{i} + 4\hat{j} - 25\hat{k}$$

$$\text{Now, } |\overline{AB} \times \overline{AC}| = \sqrt{(-36)^2 + 4^2 + (-25)^2}$$

$$= \sqrt{1296 + 16 + 625} = \sqrt{1937}$$

$$\therefore \text{Area of } \triangle ABC = \frac{1}{2} |\overline{AB} \times \overline{AC}|$$

$$= \frac{1}{2} \sqrt{1937} \text{sq. units.}$$

iv. (d) 0

Solution:

If the given points lie on the straight line,

then the points will be collinear and so area of $\triangle ABC = 0$.

$$\Rightarrow |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}| = 0$$

[\because If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the three vertices

A, B and C of $\triangle ABC$, then area of triangle

$$= \frac{1}{2} |\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|]$$

v. (b) $\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$

Solution:

$$\text{Here, } |\vec{a}| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{4 + 9 + 36}$$

$$= \sqrt{49} = 7$$

\therefore Unit vector in the direction of vector \vec{a} is

$$\hat{a} = \frac{2\hat{i} + 3\hat{j} + 6\hat{k}}{7}$$

$$= \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$